Mathematics and gesture. An odd couple: the supreme form of disembodied thinking and its crude corporeal antithesis. If mathematics engenders forms of thought which exceed that available to speech, gesture seems an atavistic return to simian calls and signals before speech.

The antithesis is puzzling. Mathematics has many affiliations -- structural, aesthetic, experiential -- with music. Music is thoroughly gestural: literally in terms of its production through the playing of instruments, but also by the sonic gestures inherent in its composition and internal dynamics. Again, mathematics is a language and language has gesture multiply folded into it. For Giorgio Agamben the relation of gesture to language is one of ‘gagging’ – it blocks speech, silences it, calls attention to language as a medium unable to express its own mediality. But, as I’ve shown elsewhere (2008), the gesture/language nexus is more complex, not least gesture’s intertwining with verbal thought, than that of a gag. In any event, before all else mathematics is a written language, an ever-expanding symbolic field of ideograms, notations and diagrams, which suggests that the proper context for relating gesture to mathematics might lie in the nature of mathematical writing.

“Gestures are disciplined mobilities of the body” (Gilles Chatelet), a formulation that gains significance from the
ongoing re-evaluation of the body’s activity within all forms of thought and cultural practice over the last few decades. We have only just begun”, the philosopher Gilles Deleuze insists, “to understand what the body is capable of”. Not least, one might add, the mathematical body.

Thus, from a cognitive science standpoint George Lakoff & Ralph Nunez (2000) talk of “embodied minds” and contend that mathematics derives from basic action schemas such as starting or stopping or continuing a motion, moving along a path, cupping hands, etc. They see these as the constituent elements, essentially corporeal metaphors of mathematical concepts, resting on a substratum derived from the schemas of containment yielding the founding abstraction of ‘sets’, of which more presently. In doing so they objectify gestures into static cognitive structures, whereas they are in reality events, materio-kinesthetic processes. Reifying them occludes their role in the symbolic dynamics of mathematical practice. In particular, and this goes to the heart of the question, it ignores a fundamental fact: gestures are inseparable from diagrams.

The obscuring of this truth is an inevitable effect of an uncritical acceptance of the set-theoretical picture we have (have been given) of mathematics since the beginning of the 20th century, from the point when Cantor’s theory of transfinite numbers and infinite sets had been accepted as legitimate mathematics, but not without paradoxical formulations, such as the set of all sets, that demanded an answer to the question: What is a set? The response was to
construct a system of axioms whose intended objects were sets and whose sole primitive relation was ‘is a member (element) of’. The axioms posited the existence of certain sets - the empty set, an infinite set -- together with ways of producing new sets from existing ones, and understood sets to be **extensional**: two sets are equal if their interiors are identical, they have exactly the same members. The axioms freed mathematicians to pursue the conceptual universalism that had made sets so attractive in the first place: namely the possibility that every mathematical entity and operation ever conceived could be defined as a certain kind of set.

A collective of mathematicians writing under the pseudonym of Bourbaki started in the 1930s to actualize this possibility, producing over the years since then thousands of pages of rigorously re-written mathematics in which every mathematical entity is a set and every mathematical argument consists of well-formed formulas in the language of sets and logical quantifiers ranging over sets.

The Bourbaki authors successfully realize the late 19th century foundational desire to identify the fundamental ‘Dinge’ in mathematics. But fuse it to an extreme and puritanical interpretation of mathematical rigor according to which nothing – no notation, definition, construction, theorem, or proof -- is allowed to refer to or invoke or rely on any attribute of the physical world which includes the mathematician’s corporeality. An interdict that forbids the drawing of diagrams but not of course the inscribing of
symbols. Contrary to normal mathematical practice, not one of their thousands of pages, Bourbaki proudly declare, contains a single diagram.

Three features of this set-based characterization of mathematics stand out.

First, objects are primary, connections between objects secondary. Although ontologically every mathematical entity whether an object (a number, group, topological space, ordered set) or a connection between objects (a function, an algebraic relation, an operation) is ultimately a set, the two are not imagined on the same level: conceptually, objects have a prior status. One defines a structure (a group, a space) as an underlying set together with an operation on its elements or subsets, then one considers relations and functions on its elements related to the operation.

Second, like the sets they’re based on, structures are conceived extensionally, completely determined by their interiors: two structures are equal if and only if their underlying sets and the sets determining their internal operations are equal.

Third, set-theoretic rigor is inherently formalistic. Bourbaki’s foundational program of re-writing ‘naive’ mathematics in correct, set-theoretic terms assures its fidelity to a severely abstract, disembodied logico-syntactical language and style of exposition. The program
is in principle unconcerned with how and why the mathematics it is in the business of re-formulating came into being: the idea behind a proof, the point of a definition, the gesture underpinning a construction, the value of a diagram, questions of intuition, aesthetics and motive, are all irrelevant, epiphenomenal to the formal content of a concept or the logical truth of a theorem.

Conceiving mathematics a la Bourbaki became the established norm for ‘correct’, rigorous presentations of the subject for the better part of the 20th century. Many still hold to it. No puzzle, then, that gesture – seemingly an even more corporeal intrusion into the rigorous, disembodied purity of mathematics than diagrams – should be invisible.

But new developments in mathematics, principally the birth and extraordinary growth of category theory, have made both the primacy of interiority and the exclusion of the diagrammatic from mathematics increasingly untenable. Among those who have challenged these two set-theoretic givens is the mathematician Gilles Chatelet. In his account of the mathematization of space, (2000) Chatelet makes gesture crucial to what he calls the “amplifying” abstraction of mathematics, pointing to its vital and constitutive role which operates precisely through its connection (as indeed it does for Merleau-Ponty) to diagrams.

Two principles organize Chatelet’s genealogy of physico-mathematical space. One, the insistence on intuition and premonition, on the ‘metaphysical’ or contemplative
dimension of mathematical thought; the other, an understanding that mathematical abstraction cannot be divorced from “sensible matter”, from the material movement and agency of bodies.

In Chatelet’s account, mathematics arises in the traffic between embodied rumination, “figures tracing contemplation”, and defined abstractions “formulae actualizing operations” (7). The vehicles that articulate the two realms and carry the traffic are metaphors. They form bridges “from premonition to certainty” (and, in doing so accomplish a “shedding [of] their skin” whereby they metamorphose into operations. (9)

Diagrams, by contrast, are figures of contemplation and rumination and can (like mandalas in Buddhist practice) focus attention, heighten awareness, and literally embody thought. For Chatelet, they are tied to gestures. “A diagram can transfix a gesture, bring it to rest, long before it curls up into a sign.” Diagrams are immobilized or frozen gestures, gestures caught in “mid-flight” that “distill action”. Unlike metaphors, whose action exhausts them, diagrams do not disappear in the act of being used. “When a diagram immobilizes a gesture in order to set down an operation, it does so by sketching a gesture that then cuts out another.” (9-10) This capacity of diagrams makes them sites for the relay and retrieval of gestures and, as such, amenable to thought experiments – imagined narratives which allow new, mathematical entities into being; entities whose conceptual salience – as an operation, a space or
whatever -- rests on the repeatability of the gestures underlying them.

For Chatelet, gesture operates on all scales, from the historical, “Gesture and problems mark an epoch” (3), to the individual level of mathematical thought where it is “crucial in our approach to the amplifying abstraction of mathematics.” (9); a mode of abstraction that cannot be captured by formal systems, which, he insists, “would like to buckle shut a grammar of gestures.” One is infused with a gesture “before knowing it”, a conception that allows him, for example, to concretize geometry as “the learning of gestures for grasping space” (2006: 42). A gesture, insofar as it is a sign is not referential “it doesn’t throw out bridges between us and things”. A gesture is not predetermined, “no algorithm controls its staging”, and it is “not substantial it gains amplitude by determining itself.” (9-10). In short, gestures are performative, in that they generate meaning through the fact and manner of their taking place; which makes the vehicles of their capture -- diagrams -- not only visible icons of similitude but also markers of invisible kinethesis. Performative but also preformative in the sense that they precede the mathematical forms -- metaphors, diagrams, symbols -- that arise in their wake.

Chatelet offers a highly schematic, unorthodox pathway of the genesis of mathematical thought from the gestures of a ruminative/contemplative body through diagrams and metaphors to symbols and formal system. With it as
background we can observe how category theory offers a
gesturo-diagrammatic characterization of mathematics
distinct from that enshrined in the language of sets.

The ur-concept of categories, is an arrow, a morphism, that
goes from one object to another. Arrows are a universal of
mathematical action. They can represent/reproduce all the
actions -- constructions, maps, transformations, operations
-- performed in set theory by functions. Making arrows
primitive has two consequences. First, they introduce
mobility -- schemas of directionality and movement,
transformation and change -- from the very beginning, in
contrast to the static structuralism of set membership.
Second, by prioritizing arrows, category theory determines
an object’s character externally: not as with sets through
internal constitution, but relationally, through the arrows
that enter and exit it. The move from interiority to
external relationality is a fundamental one, but not one
peculiar to category theory; versions of it, each
significant within their contexts, occur elsewhere. In
fact, its origin in mathematics, before set theory
instauration, is Klein’s Erlanger Programm announced in
1872 for classifying geometries by their groups of
symmetries. For Poincaré it appears as the excision of the
Ding an Sich from science: "The things themselves are not
what science can reach ... but only the relations between
things. Outside of these relations there is no knowable
reality" (1902). Likewise, the move is Saussure’s defining
structuralist turn from a referential linguistics which has
‘positive terms’ to one in which items are determined by
their differential relations to other items. Or again, it is behind Vygotsky’s, Voloshinov’s and Mead’s insistence on the individual ‘I’ originating externally, as socially not endogenously produced. A move which reappears in Lacan’s psychoanalytic understanding of unconscious interiority as the external symbolic, “as the discourse of the other”.

In short, category theory with several significant precedents and parallels thinks outside-in -- ‘sociologically’ -- about the nature of mathematical objects. Category theory’s agenda has little in common with the foundational impulse and obsession with logico-syntactical rigor that controls the Bourbaki enterprise. The difference is not a question of mathematical content (though of course the theory is established through its own theorems) but of motive, style, aesthetics, and conceptual feel. What for set theory is an edifice of structures, category theory views as a field of transformations. Its primary concern is capturing the intuitions at work in what were conceived as widely different and unconnected arenas of mathematical thought. It captures and generalizes these through a language of arrows and diagrams of arrows and diagrams of diagrams of arrows and ...

But periphrasis only goes so far. Arrows are only meaningful as actions. And like gestures they are disciplined actions. Herewith, then, a few minutes of mathematical discipline.

SLIDES 1 to 11
In summary, categories reveal mathematics as diagrammatic thought: a subtle, powerful and wide-ranging universal language with a grammar of transformations and a syntax of commuting diagrams expressing equality of morphisms from one object, category, or functor to another. Ontologically, or better ontogenetically, this gives rise to a universe of entities-in-transformation, a structuralism of becoming in place of set-theory’s fixation on being. In other words, difference, movement, verbs, and relations before identity, immutability, nouns and properties.

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A category is a collection of arrows -- morphisms -- between objects satisfying certain rules/axioms for combining arrows.

The mathematical character of an object is determined by the arrows going in and out of it.

Any mathematical entity can serve as an object:

eg
undifferentiated marks, points in space, sets, diagrams, graphs, networks, functions, relations, algebraic structures, transformations, operators, computations, partial orderings, logical propositions, topological spaces, and categories themselves.

Generally, to every species of mathematical structure there is a category whose objects have that structure and whose morphisms preserve it.

eg
category of groups and homomorphisms, topological spaces and continuous maps, vector spaces and linear transformations
A **category** consists of **morphisms** (arrows) \( f: A \rightarrow B \) between objects satisfying the following axioms

- **morphisms compose**
  if \( f: A \rightarrow B \) and \( g: B \rightarrow C \) then there exists a composite morphism \( g \circ f: A \rightarrow C \)

- **identity morphisms exist**
  every object \( A \) has an identity morphism \( \text{id}_A \) such that
  for any \( f: A \rightarrow B \)
  \( f \circ \text{id}_A = f \) and \( \text{id}_B \circ f = f \)

- **morphisms are associative**
  if \( f: A \rightarrow B \) and \( g: B \rightarrow C \) and \( h: C \rightarrow D \) then
  \( (h \circ g) \circ f = h \circ (g \circ f) \)
Objects are sets, morphisms are functions \( f : A \rightarrow B \) defined on their elements. If the sets possess structure, then morphisms preserve it.

Objects are elements of a set partially or totally ordered by the relation \( \leq \). Morphism \( x \rightarrow y \) exists when \( x \leq y \).

Eg. the subsets of a set partially ordered by inclusion, or the ordinal \( n \) of smaller ordinals totally ordered by magnitude.

A category can have just one object.

Eg. an algebraic group \((G, \cdot)\) can be construed as a category with a single object \( G \). The group identity is \( \text{id}_G \) and the group operation \( f \otimes g \) on elements of \( G \) as a set corresponds to \( f \circ g \) for morphisms \( f : G \rightarrow G \) and \( g : G \rightarrow G \).
There is no equality between objects. Equality can only exist between morphisms.

Instead, objects can be isomorphic to each other, meaning they have identical relations with all other objects in their category.

A and B are **isomorphic** if there exist morphisms 
\[ f: A \rightarrow B \] and 
\[ g: B \rightarrow A \] 

such that 
\[ g \circ f = \text{id}_A \] and \[ f \circ g = \text{id}_B \]

f and g are said to be **inverses** of each other.
construction inside a category I

An object $A$ is an **initial object** in category $C$ if there is a unique morphism: $A \rightarrow X$ for every $X$ in $C$.

$C$ may not have initial objects. If it does they are unique up to isomorphism, that is, any two initial objects are isomorphic.

Eg, 0 is an initial object in the category of an ordinal.

The concept has a **dual** (obtained by reversing arrows): an object $B$ is a **terminal object** in $C$ if there is a unique morphism: $X \rightarrow B$ for every $X$ in $C$.

Initial and terminal objects allow the formulation of an **optimum** (best or worst) construction satisfying a given condition.
The product of objects A and B is an object P with morphisms $p_A : P \rightarrow A$ and $p_B : P \rightarrow B$ such that if $P^*$ is any object with morphisms $p_A^*$ and $p_B^*$ to A and B then there exists a unique morphism: $P^* \rightarrow P$.

In other words P is a terminal object in the subcategory of candidates $P^*$.

A product if it exists is unique up to isomorphism.

The dual construction is a co-product.

The definition of a product illustrates a general phenomenon -- concepts that seem inherently set-theoretic are generalized to categories. Here the concept is the usual cartesian product $A \times B$ of two sets defined as the set of all pairs $(x, y)$ with $x$ in $A$ and $y$ in $B$. Likewise co-products generalize sums of sets.
The pullback of two morphisms \( f: A \to C \) and \( g: B \to C \) is an object \( D \) with morphisms
\[
d_A: D \to A \quad \text{and} \quad d_B: D \to B
\]
such that
\[
f \circ d_A = g \circ d_B
\]
(square DACB commutes) and
for any \( D^* \) together with \( d_A^* \) and \( d_B^* \),
there is a unique morphism \( : D^* \to D \).

In other words, \( D \) is a terminal object in the subcategory of all such candidates \( D^* \).

The concept is applicable to any number of objects with morphisms into \( C \). The pullback formulates the concept ‘the best of the worst’ (greatest lower bound of an ordered set).

The dual concept is the pushout of two morphisms
\[
f: A \to B \quad \text{and} \quad g: A \to C.
\]
‘the worst of the best’ (least upper bound of an ordered set).
Functors are like set-theory’s structure preserving functions such as homomorphisms (algebra)  continuous maps (topology) and linear transformations (vector spaces).

A functor is a structure preserving correspondence between categories it maps objects and morphisms of category $\mathbf{C}$ onto objects and morphisms of category $\mathbf{D}$ so as to preserve composition.

If $\mathbf{C}$ and $\mathbf{D}$ are categories a functor $F : \mathbf{C} \to \mathbf{D}$ is a map such that if $f : X \to Y$ in $\mathbf{C}$ then $F(f) : F(X) \to F(Y)$ in $\mathbf{D}$ and $F(g \circ f) = F(g) \circ F(f)$ (diagram in $\mathbf{D}$ commutes)
functors

comments

A simple but important eg is a forgetful functor \( F: \mathbf{C} \rightarrow \mathbf{D} \). This assigns to each object a structurally reduced version of ‘itself’.

An extreme case is a functor which forgets all structure.

Eg, let \( \mathbf{C} \) be the category of groups and group homomorphisms and \( \mathbf{D} \) the category of sets and functions.

If \( f:X \rightarrow Y \) is a homomorphism in \( \mathbf{C} \) define \( F(X) \) and \( F(Y) \) to be the underlying sets of \( X \) and \( Y \) and

\[ F(f): F(X) \rightarrow F(Y) \] to be the function \( f \) between these sets.

If \( \mathbf{C}^+ \) is a category whose objects \( \mathbf{C}, \mathbf{D} \) etc are categories then morphisms \( F: \mathbf{C} \rightarrow \mathbf{D} \) in \( \mathbf{C}^+ \) are functors.

Functors generalize homomorphisms.

For one-object categories the two notions coincide. Thus, if \( G \) and \( H \) are groups construed as one-object categories then functors \( F: G \rightarrow H \) are precisely group homomorphisms.
There are in general many functors between two categories. One wants to be able to compare them. The concept of a natural transformation accomplishes this.

Suppose $F: \textbf{C} \rightarrow \textbf{D}$ and $G: \textbf{C} \rightarrow \textbf{D}$ are functors.

A **natural transformation** $\mu$ is a morphism $\mu: F \rightarrow G$ comprising a family of morphisms $\mu_X: F(X) \rightarrow G(X)$ and $\mu_Y: F(Y) \rightarrow G(Y)$ in $\textbf{D}$ for each $X$ and $Y$ in $\textbf{C}$ such that

$$G(f) \circ \mu_X = \mu_Y \circ F(f)$$

(diagram in $\textbf{D}$ commutes for each $X$ and $Y$)
Natural transformations first arose in the study of topological spaces. One studies topological spaces by assigning group structures -- e.g., homology and homotopy groups -- to spaces, and proving theorems about the groups. Such assignments are functors. It was to define functors and understand relations between them that Samuel Eilenberg and Saunders MacLane were led in the 1940s to invent the notion of a category.

An essential use of natural transformations occurs when the functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ go in opposite directions. By considering pairs of natural transformations $\mu: F \circ G \to \text{id}_D$ and $\pi: \text{id}_C \to G \circ F$ satisfying a certain optimality condition, one obtains the all-important concept of adjoint functors which formulates the idea of $F$ and $G$ being conceptual inverses of each other.